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Compact finite difference method for the fractional diffusion equation

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ABSTRACT

High-order compact finite difference scheme for solving one-dimensional fractional diffusion equation is considered in this paper. After approximating the second-order derivative with respect to space by the compact finite difference, we use the Grünwald–Letnikov discretization of the Riemann–Liouville derivative to obtain a fully discrete implicit scheme. We analyze the local truncation error and discuss the stability using the Fourier method, then we prove that the compact finite difference scheme converges with the spatial accuracy of fourth order using matrix analysis. Numerical results are provided to verify the accuracy and efficiency of the proposed algorithm.

1. Introduction

Fractional differential equations (FDEs) have attracted increasing attention because they have applications in various fields of science and engineering. For example, they can describe many physical and chemical processes, biological systems, etc. The main physical purpose for investigating diffusion equations of fractional order is to describe phenomena of anomalous diffusion in transport processes through complex and/or disordered systems including fractal media, and fractional kinetic equations have proved particularly useful in the context of anomalous slow diffusion, see, for example, the excellent review paper [1]. An overview of the basic theory of fractional differentiation, fractional-order differential equations, methods of their solution and applications can be found in the book [2].

There have been several numerical methods proposed for solving the space and/or time FDEs up to now. Lynch et al. [3] developed two numerical schemes, one explicit and another one semi-implicit, for solving the transport problem with anomalous diffusion modeled by a partial differential equation of fractional order. Meerschaert and Tadjeran [4] studied the one-dimensional radial flow model, they found that the fractional derivative describes more accurately the early arrival that cannot be explained by the classical advection–dispersion equations and they presented an implicit Euler method, based on a

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modified Grünwald approximation for the fractional derivative. Langlands and Henry [5] investigated the fractional diffusion equation and proposed an implicit numerical scheme but without the global accuracy of the numerical scheme. Yuste and Acedo [6] proposed an explicit finite difference method and analyzed the condition for stability for the fractional subdiffusion equation. Recently, Chen et al. [7] employed a difference approximation scheme for solving the fractional diffusion equation, analyzed the stability and the accuracy by the Fourier method, and Zhuang et al. [8] investigated the stability and convergence of an implicit numerical method by the energy method. Both implicit and explicit finite difference methods for fractional reaction-subdiffusion equations were given in Chen et al. [9], and an implicit numerical method and new analytical techniques were introduced for the modified anomalous subdiffusion equation with a nonlinear source term in Liu et al. [10]. The fractional heat equations were discussed in [11] and matrix approach to discrete fractional calculus were given in [12,13]. For the theoretical solutions, the velocity field for the fractional anomalous diffusion caused by a plate moving impulsively in its own plane was studied in Xu and Tan [14].

To the author's knowledge, we have not seen a scheme that is convergent with order higher than two for the space variable. Therefore, it is interesting to discuss high-order numerical methods for FDEs. Because compact finite difference schemes have the advantages of the fourth-order accuracy to approximate the second-order derivatives and keeping the desirable tridiagonal nature of the finite-difference equations, they have been discussed by Hirsch [15] and Lele [16]. The compact finite difference approximation for the second order space derivative is explained in paper [17] with applications for reaction-diffusion problems. Recently, Cui [18] considered the compact finite difference scheme to the generalized one-dimensional sine-Gordon equation with error analysis.

The main purpose of the this paper is to solve the fractional diffusion problem using the compact difference scheme and give the stability and convergence analysis. Previous methods for fractional subdiffusion problems have been limited to second-order accuracy in space. Compact scheme is a high-order method and the coefficient matrix of the linear system of equations of the unknowns is tridiagonal and can be easily solved by the Thomas algorithm. Because implicit schemes have better stability properties than the corresponding explicit ones, we consider the implicit compact difference approximation scheme in our paper. As pointed in the paper [4], the general preference for the Crank–Nicolson scheme for the classical partial differential equations is that it is second-order accurate in time. However, the Grünwald–Letnikov estimates are only first order accurate, and therefore in the paper we give the backward Euler scheme. To increase the accuracy of time, we can use the Richardson extrapolation technique as proposed in paper [8]. The model problem considered here is the one-dimensional fractional diffusion equation describing subdiffusive phenomena with a non-homogeneous term [1,6],

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[K_\gamma \frac{\partial^2 u}{\partial x^2}\right] + f(x,t), \quad x \in (L_0, L_1), \quad 0 < t < T, \tag{1}$$

where K_γ is the generalized diffusion constant, and ${}_0D_t^{1-\gamma}u(0<\gamma<1)$ denotes the Riemann–Liouville fractional derivative of order $1-\gamma$ of the function $v(x,t)$, defined by [2], i.e.,

$${}_0D_t^{1-\gamma}v = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{v(x,\tau)}{(t-\tau)^{1-\gamma}}d\tau. \tag{2}$$

Note that in the limit $\gamma \to 1$, the FDE (1) reduces to Fick's second law, and (1) corresponds to the ordinary (or Brownian) diffusion equation. The initial condition for (1) is

$$u(x,0) = w(x) \tag{3}$$

and the Dirichlet boundary condition for (1) is given by (with $\varphi(0) = w(L_0)$ and $\psi(0) = w(L_1)$ for consistency)

$$u(L_0,t) = \varphi(t), \quad u(L_1,t) = \psi(t), \quad t \geq 0. \tag{4}$$

The paper is organized as follows: in Section 2, we present an implicit compact difference scheme. We approximate the second-order derivative with respect to space by the compact finite difference, then we use the Grünwald–Letnikov discretization for the approximation of the time fractional derivative. In Sections 3 the matrix form for the difference scheme is given, and the solvability for the linear system of equations is discussed. In Section 4 we give the local truncation error, investigate the stability by the Fourier method, and discuss the convergence of the scheme using matrix analysis, follow the paper by Zhao et al. [19]. We prove that the scheme is unconditionally stable for all γ in the range $0 < \gamma < 1$, derive the global accuracy and prove the convergence of the scheme. Finally, some numerical results are provided in Section 5, they are in agreement with our theoretical analysis. The paper concludes with a summary in Section 6.

In this paper, the symbol C is a generic positive constant, it may take different value at different places. We use the "empty sum" convention $\sum_{l=p}^{q} v^l = 0$ for $q < p$.

2. Compact finite difference scheme

2.1. Partition and the solution vector

For the numerical solution of the problem above we introduce a uniform grid of mesh points (x_j, t_k), with $x_j = L_0 + jh, j = 0, 1, \ldots, M$ and $t_k = k\tau, k = 0, 1, \ldots, N$ where M and N are positive integers, $h = (L_1 - L_0)/M$ is the mesh-width in x and $\tau = T/N$ the time step. The theoretical solution u at the point (x_j, t_k) is denoted by u_j^k. The solution of an approximating difference scheme at the same point will be denoted by U_j^k. We denote the exact solution vector of order N by $\mathbf{u}^k = \mathbf{u}(t_k) = (u_1^k, \dots, u_{M-1}^k)^T$ and the approximate solution vector $\mathbf{U}^k = \mathbf{U}(t_k) = (U_1^k, \dots, U_{M-1}^k)^T$.

2.2. Derivation of the numerical scheme

We give the numerical solution for the model problem 1, 3 and 4. As the familiar central difference quotient defined by

$$\frac{1}{h^2}\delta_x^2 u_j \equiv \frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}) = \left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j h^2 + \mathcal{O}(h^4) \tag{5}$$

gives only second-order approximation to u_{xx}, we can use the compact finite difference operator instead, and maintaining the three-point stencil. Introduce the central difference operator $\delta_x u_j = u_{j+1/2} - u_{j-1/2}$, then higher-order finite difference operators are derived from the approximation ((1–69) and (1–70) in [20])

$$\begin{aligned}\left(\frac{\partial^2 u}{\partial x^2}\right)_j &= \left[\frac{2}{h}\sinh^{-1}\frac{\delta_x}{2}\right]^2 u_j = \frac{1}{h^2}\left[\delta_x - \frac{1^2}{2^2\cdot 3!}\delta_x^3 + \frac{1^2\cdot 3^2}{2^4\cdot 5!}\delta_x^5 - \frac{1^2\cdot 3^2\cdot 5^2}{2^6\cdot 7!}\delta_x^7 + \cdots\right]^2 u_j \\ &= \frac{1}{h^2}\left(\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{90}\delta_x^6 - \frac{1}{560}\delta_x^8 + \cdots\right)u_j.\end{aligned}$$

That is, we can use

$$\begin{aligned}\frac{\delta_x^2}{h^2(1+\frac{1}{12}\delta_x^2)}u_j^k &= \frac{1}{h^2}\left(\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{144}\delta_x^6 - \frac{1}{1728}\delta_x^8 + \cdots\right)u_j^k = \frac{\partial^2 u}{\partial x^2}\Big|_j^k - \frac{1}{240h^2}\delta_x^6 u_j^k + \mathcal{O}(h^6) \\ &= \frac{\partial^2 u}{\partial x^2}\Big|_j^k - \frac{1}{240}\frac{\partial^4 u}{\partial x^4}\Big|_j^k h^4 + \mathcal{O}(h^6)\end{aligned} \tag{6}$$

to keep the fourth-order accuracy and the tridiagonal nature of the schemes.

Using the relationship between the Grünwald–Letnikov formula and the Riemann–Liouville fractional derivatives [2], we can approximate the fractional derivative by

$${}_0D_t^{1-\gamma}f(t) = \frac{1}{\tau^{1-\gamma}}\sum_{k=0}^{[t/\tau]}\omega_k^{(1-\gamma)}f(t-k\tau) + \mathcal{O}(\tau^p), \tag{7}$$

where $\omega_k^{(1-\gamma)}$ are the coefficients of the generating function, that is, $\omega(z,\alpha) = \sum_{k=0}^{\infty}\omega_k^{(\alpha)}z^k$. We will discuss the case for $\omega(z,\alpha) = (1-z)^\alpha$, and thus $p = 1$. In this case, these coefficients are $\omega_0^{(\alpha)} = 1$ and $\omega_k^{(\alpha)} = (-1)^k\binom{\alpha}{k} = (-1)^k\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ for $k \geq 1$ and can be evaluated recursively,

$$\omega_0^{(\alpha)} = 1, \quad \omega_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right)\omega_{k-1}^{(\alpha)}, \quad k \geq 1. \tag{8}$$

See, for example, papers [6,21] for the details. For any function $v(x,t)$, we let $v_j^k = v(x_j, t_k)$ and for the convenience of notations, we put these coefficients $\lambda_l \equiv \omega_l^{(1-\gamma)} = (-1)^l\binom{1-\gamma}{l}, l = 0, 1, \dots, k$. Then the implicit compact finite difference method for 1, 3 and 4 is given as follows:

$$\begin{cases}\frac{U_j^k - U_j^{k-1}}{\tau} = K_\gamma\frac{\tau^{\gamma-1}}{h^2}\sum_{l=0}^{k}\lambda_l\frac{\delta_x^2}{1+\frac{1}{12}\delta_x^2}U_j^{k-l} + f_j^k, \quad j = 1,2,\dots,M-1, \quad k = 1,2,\dots,N, \\ U_j^0 = w(x_j), \quad j = 1,2,\dots,M-1, \\ U_0^k = \varphi(t_k), \quad U_M^k = \psi(t_k), \quad k = 0,1,\dots,N.\end{cases} \tag{9}$$

Introduce the scaling parameter $\mu = K_\gamma\frac{\tau^\gamma}{h^2}$ and multiply both sides of (9) by the operator $(1 + \frac{1}{12}\delta_x^2)$, after rearranging the terms and noting that $\lambda_0 = 1$, we have

$$\begin{cases}(1 + (\frac{1}{12} - \mu)\delta_x^2)U_j^k - (1 + \frac{1}{12}\delta_x^2)U_j^{k-1} = \mu\sum_{l=1}^{k}\lambda_l\delta_x^2 U_j^{k-l} + \tau(1 + \frac{1}{12}\delta_x^2)f_j^k, \quad 1 \leq j \leq M-1, \quad 1 \leq k \leq N, \\ U_j^0 = w(x_j), \quad j = 1,2,\dots,M-1, \\ U_0^k = \varphi(t_k), \quad U_M^k = \psi(t_k), \quad k = 0,1,\dots,N.\end{cases} \tag{10}$$

On the mesh points $(x_j, t_k), j = 1,2,\dots,M-1$, we get

$$\begin{cases} \left(\frac{1}{12}-\mu\right)U_{j-1}^1+\left(\frac{5}{6}+2\mu\right)U_j^1+\left(\frac{1}{12}-\mu\right)U_{j+1}^1=\left(\frac{1}{12}+\mu\lambda_1\right)U_{j-1}^0+\left(\frac{5}{6}-2\mu\lambda_1\right)U_j^0+\left(\frac{1}{12}+\mu\lambda_1\right)U_{j+1}^0+\tau\left(\frac{1}{12}f_{j-1}^1+\frac{5}{6}f_j^1+\frac{1}{12}f_{j+1}^1\right), \\ \left(\frac{1}{12}-\mu\right)U_{j-1}^k+\left(\frac{5}{6}+2\mu\right)U_j^k+\left(\frac{1}{12}-\mu\right)U_{j+1}^k \\ \quad =\left(\frac{1}{12}+\mu\lambda_1\right)U_{j-1}^{k-1}+\left(\frac{5}{6}-2\mu\lambda_1\right)U_j^{k-1}+\left(\frac{1}{12}+\mu\lambda_1\right)U_{j+1}^{k-1}+\mu\sum_{l=0}^{k-2}\lambda_{k-l}(U_{j-1}^l-2U_j^l+U_{j+1}^l)+\tau\left(\frac{1}{12}f_{j-1}^k+\frac{5}{6}f_j^k+\frac{1}{12}f_{j+1}^k\right),\ 2\le k\le N, \\ U_j^0=w(x_j),\ U_0^k=\varphi(t_k),\ U_M^k=\psi(t_k),\ k=0,1,\ldots,N. \end{cases} \tag{11}$$

And we can see that the coefficients for U_{j-1}^0, U_j^0 and U_{j+1}^0 are different for $k=1$ and $k\ge 2$, and this makes the matrix B_0 different for $k=1$ and $k\ge 2$ below. Use the "empty sum" convention, we can write (11) simply as

$$\begin{aligned} &\left(\frac{1}{12}-\mu\right)U_{j-1}^k+\left(\frac{5}{6}+2\mu\right)U_j^k+\left(\frac{1}{12}-\mu\right)U_{j+1}^k \\ &\quad=\left(\frac{1}{12}+\mu\lambda_1\right)U_{j-1}^{k-1}+\left(\frac{5}{6}-2\mu\lambda_1\right)U_j^{k-1}+\left(\frac{1}{12}+\mu\lambda_1\right)U_{j+1}^{k-1}+\mu\sum_{l=0}^{k-2}\lambda_{k-l}(U_{j-1}^l-2U_j^l+U_{j+1}^l) \\ &\qquad+\tau\left(\frac{1}{12}f_{j-1}^k+\frac{5}{6}f_j^k+\frac{1}{12}f_{j+1}^k\right), \quad 1\le j\le M-1, \quad 1\le k\le N. \end{aligned}$$

3. Matrix form of the numerical scheme

Multiply the compact implicit difference approximation scheme (10) or (11) by a common factor 12, we can give the matrix form of the scheme by

$$\begin{cases} A\mathbf{U}^1=\widetilde{B}_0\mathbf{U}^0+\mathbf{F}^1, \\ A\mathbf{U}^k=\sum_{l=0}^{k-1}B_l\mathbf{U}^l+\mathbf{F}^k, \quad k=2,3,\ldots,N, \end{cases} \tag{12}$$

where the tridiagonal matrices in (12) are given by

$$A=\begin{pmatrix} 10+24\mu & 1-12\mu & & & \\ 1-12\mu & 10+24\mu & 1-12\mu & & \\ & \ddots & \ddots & \ddots & \\ & & 1-12\mu & 10+24\mu & 1-12\mu \\ & & & 1-12\mu & 10+24\mu \end{pmatrix}_{(M-1)\times(M-1)},$$

$$\widetilde{B}_0=\begin{pmatrix} 10-24\mu\lambda_1 & 1+12\mu\lambda_1 & & & \\ 1+12\mu\lambda_1 & 10-24\mu\lambda_1 & 1+12\mu\lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1+12\mu\lambda_1 & 10-24\mu\lambda_1 & 1+12\mu\lambda_1 \\ & & & 1+12\mu\lambda_1 & 10-24\mu\lambda_1 \end{pmatrix}_{(M-1)\times(M-1)},$$

$$B_l=12\mu\lambda_{k-l}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}_{(M-1)\times(M-1)}, \quad l=0,1,\ldots,k-2,$$

$$B_{k-1}=\widetilde{B}_0, \quad k\ge 2,$$

and finally, the column vectors in $R^{(M-1)}$ are given by

$$\mathbf{F}^1=\begin{pmatrix} (1+12\mu\lambda_1)U_0^0-(1-12\mu)U_0^1+\tau(f_0^1+10f_1^1+f_2^1) \\ \tau(f_1^1+10f_2^1+f_3^1) \\ \vdots \\ \tau(f_{M-3}^1+10f_{M-2}^1+f_{M-1}^1) \\ (1+12\mu\lambda_1)U_M^0-(1-12\mu)U_M^1+\tau(f_{M-2}^1+10f_{M-1}^1+f_M^1) \end{pmatrix},$$

$$\mathbf{F}^k = \begin{pmatrix} 12\mu\sum_{l=0}^{k-2}\lambda_{k-l}U_0^l + (1+12\mu\lambda_1)U_0^{k-1} - (1-12\mu)U_0^k + \tau(f_0^k + 10f_1^k + f_2^k) \\ \tau(f_1^k + 10f_2^k + f_3^k) \\ \vdots \\ \tau(f_{M-3}^k + 10f_{M-2}^k + f_{M-1}^k) \\ 12\mu\sum_{l=0}^{k-2}\lambda_{k-l}U_M^l + (1+12\mu\lambda_1)U_M^{k-1} - (1-12\mu)U_M^k + \tau(f_{M-2}^k + 10f_{M-1}^k + f_M^k) \end{pmatrix}, \quad k \geq 2.$$

For the solvability of the scheme we have

Theorem 1. *The difference system (10) has a unique solution.*

Proof. Because for any $\mu = K_\gamma \frac{\tau^\gamma}{h^2} > 0$, the coefficient matrix A for the difference equations is strictly diagonally dominant. Consequently, the matrix A is nonsingular, thus it is invertible. Therefore, the solution of our compact difference scheme exists and is unique. □

4. Theoretical analysis of the compact finite difference scheme

4.1. The local truncation error

We give the local truncation error of our scheme. First, we can use (7) to obtain

$$_0D_t^{1-\gamma}f(t) = \tau^{\gamma-1}\sum_{l=0}^{[t/\tau]}\lambda_l f(t-l\tau) + \mathcal{O}(\tau). \tag{13}$$

Therefore, let $t = k\tau$ and $f(t) \equiv 1$, we get $\frac{(k\tau)^{\gamma-1}}{\Gamma(\gamma)} = \frac{t^{\gamma-1}}{\Gamma(\gamma)} = {}_0D_t^{\gamma-1}1 = \tau^{\gamma-1}\sum_{l=0}^k\lambda_l + \mathcal{O}(\tau)$. That is, we obtain the following identity, $\tau^{\gamma-1}\sum_{l=0}^k\lambda_l = \frac{t^{\gamma-1}}{\Gamma(\gamma)} + \mathcal{O}(\tau)$. [1] Hence the local truncation error of the scheme (10) is (note that $0 < \gamma < 1$)

$$\begin{aligned} R_j^k &= \frac{u_j^k - u_j^{k-1}}{\tau} - K_\gamma\frac{\tau^{\gamma-1}}{h^2}\sum_{l=0}^k\lambda_l\frac{\delta_x^2}{1+\frac{1}{12}\delta_x^2}u_j^{k-l} - f_j^k \\ &= \left(\frac{u_j^k - u_j^{k-1}}{\tau} - \frac{\partial u}{\partial t}|_j^k\right) + \left({}_0D_t^{1-\gamma}\left[K_\gamma\frac{\partial^2 u}{\partial x^2}\right]\Big|_j^k - \tau^{\gamma-1}\sum_{l=0}^k\lambda_l K_\gamma\frac{\partial^2 u}{\partial x^2}|_j^{k-l}\right) + K_\gamma\tau^{\gamma-1} \\ &\quad\times\sum_{l=0}^k\lambda_l\left(\frac{\partial^2 u}{\partial x^2}\Big|_j^{k-l} - \frac{\delta_x^2}{h^2(1+\frac{1}{12}\delta_x^2)}u_j^{k-l}\right) \\ &= \mathcal{O}(\tau) + K_\gamma\tau^{\gamma-1}\sum_{l=0}^k\lambda_l\left(-\frac{1}{240}\frac{\partial^4 u}{\partial x^4}|_j^{k-l}h^4 + \cdots\right) = \mathcal{O}(\tau) + \mathcal{O}((k\tau)^{\gamma-1}h^4) = \mathcal{O}((k^{\gamma-1}\tau^{\gamma-1}+1)(\tau+h^4)). \end{aligned} \tag{14}$$

4.2. Stability

Stability analysis of the difference approximation scheme can be discussed by the Fourier method, as given in paper [7]. Let $\widetilde{U}_j^k$ be the approximate solution of (10), and define

$$\rho_j^k = U_j^k - \widetilde{U}_j^k, \quad 1 \leq j \leq M-1, \quad 0 \leq k \leq N$$

with corresponding vector

$$\rho^k = (\rho_1^k, \rho_2^k, \ldots, \rho_{M-1}^k)^T.$$

Then we have

$$\left(1 + \left(\frac{1}{12} - \mu\right)\delta_x^2\right)\rho_j^k - \left(1 + \frac{1}{12}\delta_x^2\right)\rho_j^{k-1} = \mu\sum_{l=1}^k\lambda_l\delta_x^2\rho_j^{k-l}, \quad 1 \leq j \leq M-1,\ 1 \leq k \leq N. \tag{15}$$

We expand ρ^k into a piecewise constant function, that is, for $k = 0, 1, \ldots, N$, we define the grid function

[1] Note that Lemma 2 in [7] only holds for $k\tau = 1$. Therefore, we cannot obtain $R_j^k = \mathcal{O}(\tau + h^2)$ for all k. The local truncation error given here is similar to (2.6) in paper [8] or (24) in paper [10].

$$\rho^k(x) = \begin{cases} 0, & L_0 \le x \le L_0 + \frac{h}{2}, \\ \rho_j^k, & x_j - \frac{h}{2} < x \le x_j + \frac{h}{2}, \quad j = 1, 2, \ldots, M-1, \\ 0, & L_1 - \frac{h}{2} < x \le L_1. \end{cases}$$

We can expand $\rho^k(x)$ in a Fourier series

$$\rho^k(x) = \frac{1}{\sqrt{L_1 - L_0}} \sum_{m=-\infty}^{\infty} c_k(m) e^{\mathrm{i}2\pi m(x-L_0)/(L_1-L_0)}, \quad 0 \le k \le N.$$

For all $\mathbf{w} = (w_1, \cdots, w_{M-1})^T \in R^{M-1}$, we define a discrete l^2 norm by $\|\mathbf{w}\|_{l^2} = \left(h\sum_{j=1}^{M-1} w_j^2\right)^{1/2}$. It is easy to know that $\|\mathbf{w}\|_{l^2}^2 = (h\mathbf{w}, \mathbf{w})$ where $(\cdot, \cdot)$ stands for the inner product. Then we have

Lemma 1. *The discrete Fourier coefficients are*

$$c_k(m) = \frac{1}{\sqrt{L_1 - L_0}} \int_{L_0}^{L_1} e^{-\mathrm{i}2\pi m(\xi-L_0)/(L_1-L_0)} \rho^k(\xi) d\xi$$

and we have Parseval's equality for the discrete Fourier transform, that is,

$$\int_{L_0}^{L_1} |\rho^k(x)|^2 dx = \sum_{j=1}^{M-1} h|\rho_j^k|^2 = \|\rho^k\|_{l^2}^2 = \sum_{m=-\infty}^{\infty} |c_k(m)|^2.$$

Proof. The proof can be given similarly to those of Proposition 3.1.2 and 3.1.3 in book [22]. In fact, we have

$$\begin{aligned} \frac{1}{\sqrt{L_1 - L_0}} \int_{L_0}^{L_1} e^{-\mathrm{i}2\pi l(\xi-L_0)/(L_1-L_0)} \rho^k(\xi) d\xi &= \frac{1}{\sqrt{L_1 - L_0}} \int_{L_0}^{L_1} e^{-\mathrm{i}2\pi l(\xi-L_0)/(L_1-L_0)} \frac{1}{\sqrt{L_1 - L_0}} \sum_{m=-\infty}^{\infty} c_k(m) e^{\mathrm{i}2\pi m(\xi-L_0)/(L_1-L_0)} d\xi \\ &= \frac{1}{L_1 - L_0} \sum_{m=-\infty}^{\infty} c_k(m) \int_{L_0}^{L_1} e^{\mathrm{i}2\pi(m-l)(\xi-L_0)/(L_1-L_0)} d\xi \\ &= \frac{1}{L_1 - L_0} \sum_{\substack{m=-\infty \\ m \ne l}}^{\infty} c_k(m) \left[\frac{(L_1 - L_0)}{\mathrm{i}2\pi(m-l)} e^{\mathrm{i}2\pi(m-l)(\xi-L_0)/(L_1-L_0)} \right]\Big|_{L_0}^{L_1} + \frac{1}{L_1 - L_0} c_k(l) \int_{L_0}^{L_1} d\xi \\ &= \frac{1}{L_1 - L_0} \sum_{\substack{m=-\infty \\ m \ne l}}^{\infty} c_k(m) \frac{(L_1 - L_0)}{\mathrm{i}2\pi(m-l)} [e^{\mathrm{i}2\pi(m-l)} - 1] + c_k(l) = c_k(l) \end{aligned}$$

and

$$\begin{aligned} \int_{L_0}^{L_1} |\rho^k(x)|^2 dx &= \int_{L_0}^{L_1} \overline{\rho^k(x)} \rho^k(x) dx = \int_{L_0}^{L_1} \overline{\rho^k(x)} \frac{1}{\sqrt{L_1 - L_0}} \sum_{m=-\infty}^{\infty} c_k(m) e^{\mathrm{i}2\pi m(x-L_0)/(L_1-L_0)} dx \\ &= \frac{1}{\sqrt{L_1 - L_0}} \sum_{m=-\infty}^{\infty} c_k(m) \int_{L_0}^{L_1} \overline{\rho^k(x)} e^{\mathrm{i}2\pi m(x-L_0)/(L_1-L_0)} dx = \sum_{m=-\infty}^{\infty} c_k(m) \overline{\frac{1}{\sqrt{L_1 - L_0}} \int_{L_0}^{L_1} e^{-\mathrm{i}2\pi m(x-L_0)/(L_1-L_0)} \rho^k(x) dx} \\ &= \sum_{m=-\infty}^{\infty} c_k(m) \overline{c_k(m)} = \sum_{m=-\infty}^{\infty} |c_k(m)|^2 \end{aligned}$$

This finishes the proof. $\square$

Based on the above analysis, we can suppose that the solution of (15) has the following form

$$\phi_j^k = \frac{1}{\sqrt{L_1 - L_0}} d_k e^{\mathrm{i}\sigma jh},$$

where $\sigma = 2\pi/(L_1 - L_0)$. Substituting the above expression into (15), we obtain

$$(1 + 4\mu \sin^2 \frac{\sigma h}{2}) d_1 = \left(1 - 4\mu\lambda_1 \sin^2 \frac{\sigma h}{2}\right) d_0$$

for $k = 1$ and for $2 \le k \le N$,

$$\left(1 + 4\mu \sin^2 \frac{\sigma h}{2}\right) d_k = \left(1 - 4\mu\lambda_1 \sin^2 \frac{\sigma h}{2}\right) d_{k-1} - 4\mu \sin^2 \frac{\sigma h}{2} \sum_{l=0}^{k-2} \lambda_{k-l} d_l.$$

Consequently,

$$\begin{cases} d_1 = \frac{1-4\mu\lambda_1\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} d_0, \\ d_k = \frac{1-4\mu\lambda_1\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} d_{k-1} - \frac{4\mu\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} \sum_{l=0}^{k-2}\lambda_{k-l}d_l, \quad 2 \le k \le N. \end{cases} \tag{16}$$

Note that $0 < \gamma < 1$, then the following lemma holds (see, for example, [7])[2].

Lemma 2. *The coefficients* $\lambda_l (l = 0, 1, \ldots)$ *satisfy*

(1) $\lambda_0 = 1, \quad lambda_1 = \gamma - 1, \quad \lambda_l < 0, \quad l = 1, 2, \ldots$
(2) $\sum_{l=0}^{\infty}\lambda_l = 0$, and for all $n \ge 1, \quad -\sum_{l=1}^{n}\lambda_l < 1$.

Thus we get

Lemma 3. *Suppose that* $d_k (1 \le k \le N)$ *are defined by (16), then for* $0 < \gamma < 1$, *we have*

$$|d_k| \le |d_0|, \quad k = 1, 2, \ldots, N.$$

Proof. We will use mathematical induction to complete the proof. For $k = 1$, from the first equation in (16) we have

$$|d_1| \le \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}|d_0|.$$

Noticing that $0 < \gamma < 1$, we obtain $|d_1| \le |d_0|$. Suppose that we have proved that $|d_n| \le |d_0|, 1 \le n \le k-1$, with the second equation in (16) and Lemma 2, we have

$$\begin{aligned} |d_k| &\le \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}|d_{k-1}| + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}\sum_{l=0}^{k-2}|\lambda_{k-l}||d_l| \\ &\le \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}|d_0| + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}\left(\sum_{l=0}^{k-1}|\lambda_{k-l}| - |\lambda_1|\right)|d_0| \\ &\le \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}|d_0| + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}\left(-\sum_{l=1}^{k}\lambda_l - (1-\gamma)\right)|d_0| \\ &\le \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}|d_0| + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}(1 - (1-\gamma))|d_0| = |d_0|. \end{aligned}$$

That is, $|d_n| \le |d_0|$ holds true for $n = k$, hence the proof is completed. □

For the stability of the scheme, we have

Theorem 2. *The compact difference scheme defined by (10) is unconditionally stable for* $0 < \gamma < 1$.

Proof. Suppose that $\widetilde{\mathbf{U}}^k$ is the approximate solution of (10). Applying Lemma 3 and Parseval's equality, we obtain

$$\begin{aligned} \|\mathbf{U}^k - \tilde{\mathbf{U}}^k\|_{l^2}^2 = \|\rho^k\|_{l^2}^2 = \sum_{j=1}^{M-1} h|\rho_j^k|^2 = \frac{h}{L_1 - L_0}\sum_{j=1}^{M-1}|d_k e^{\mathrm{i}\sigma jh}|^2 = \frac{h}{L_1 - L_0}\sum_{j=1}^{M-1}|d_k|^2 \\ \le \frac{h}{L_1 - L_0}\sum_{j=1}^{M-1}|d_0|^2 = \frac{h}{L_1 - L_0}\sum_{j=1}^{M-1}|d_0 e^{\mathrm{i}\sigma jh}|^2 = \|\rho^0\|_{l^2}^2 = \|\mathbf{U}^0 - \tilde{\mathbf{U}}^0\|_{l^2}^2, \quad k = 1, 2, \ldots, N \end{aligned}$$

which proves that scheme (10) is unconditionally stable. □

4.3. Convergence

For integer order partial differential equations, by the Lax equivalence theorem it follows then that stability and consistency imply convergence. For the fractional order ones, Lubich [21] proved that the numerical approximation of fractional integrals is convergent of order p if and only if it is stable and consistent of order p. We will prove that the compact finite

[2] There is a typo in [7], $\sum_{l=0}^{\infty}\lambda_l$ should be zero, as $\lambda_l (l = 0, 1, \ldots)$ are the coefficients of the generating function $(1-z)^{1-\gamma}$.

difference scheme converges with the spatial accuracy of fourth order. For this purpose, we use the following discrete Gronwall Lemma 1.4.2 in [23]).

Lemma 4. *Assume that k_n is a non-negative sequence, and that the sequence ϕ_n satisfies*

$$\begin{cases} \phi_0 \le g_0, \\ \phi_n \le g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s \phi_s, \quad n \ge 1. \end{cases}$$

Then ϕ_n satisfies

$$\begin{cases} \phi_1 \le g_0(1+k_0) + p_0, \\ \phi_n \le g_0 \prod_{s=0}^{n-1}(1+k_s) + \sum_{s=0}^{n-2} p_s \prod_{r=s+1}^{n-1}(1+k_r) + p_{n-1}, \quad n \ge 2. \end{cases}$$

Moreover, if $g_0 \ge 0$ and $p_n \ge 0$ for $n \ge 0$, it follows

$$\phi_n \le (g_0 + \sum_{s=0}^{n-1} p_s) \exp(\sum_{s=0}^{n-1} k_s), \quad n \ge 1.$$

Now we have

Theorem 3. *The compact difference scheme defined by (10) is first order accurate in time and fourth accurate in the space variable, i.e.,*

$$\|\mathbf{u}^k - \mathbf{U}^k\|_{l^2} \le C(\tau + h^4), \quad k = 1, 2, \dots, N.$$

Proof. We conclude from (14) that

$$|R_j^k| \le C(k^{\gamma-1}\tau^{\gamma-1} + 1)(\tau + h^4), \quad 1 \le k \le N. \tag{17}$$

Let $e_j^k = u_j^k - U_j^k, 1 \le j \le M-1, \mathbf{e}^k = (e_1^k, e_2^k, \dots, e_{M-1}^k)^T, \mathbf{R}^k = (R_1^k, R_2^k, \dots, R_{M-1}^k)^T, 1 \le k \le N$, then we have

$$\left(1 + \left(\frac{1}{12} - \mu\right)\delta_x^2\right)u_j^k - \left(1 + \frac{1}{12}\delta_x^2\right)u_j^{k-1} = \mu \sum_{l=1}^{k} \lambda_l \delta_x^2 u_j^{k-l} + \tau\left(1 + \frac{1}{12}\delta_x^2\right)f_j^k + \tau\left(1 + \frac{1}{12}\delta_x^2\right)R_j^k,$$
$$1 \le j \le M-1, \quad 1 \le k \le N,$$

and

$$\left|\left(1 + \frac{1}{12}\delta_x^2\right)R_j^k\right| \le \left|\frac{1}{12}R_{j-1}^k\right| + \left|\frac{5}{6}R_j^k\right| + \left|\frac{1}{12}R_{j+1}^k\right| \le C(k^{\gamma-1}\tau^{\gamma-1} + 1)(\tau + h^4).$$

Let $\widetilde{\mathbf{R}}^k = \left((1 + \frac{1}{12}\delta_x^2)R_1^k, (1 + \frac{1}{12}\delta_x^2)R_2^k, \cdots, (1 + \frac{1}{12}\delta_x^2)R_{M-1}^k\right)^T$, then we have

$$\begin{cases} A\mathbf{u}^1 = \widetilde{B}_0\mathbf{u}^0 + \mathbf{F}^1 + \tau\widetilde{\mathbf{R}}^1, \\ A\mathbf{u}^k = \sum_{l=0}^{k-1} B_l \mathbf{u}^l + \mathbf{F}^k + \tau\widetilde{\mathbf{R}}^k, \quad k = 2, 3, \dots, N. \end{cases} \tag{18}$$

Note that $\mathbf{e}^0 = 0$, subtract (18) from (12), we get

$$\begin{cases} A\mathbf{e}^1 = \tau\widetilde{\mathbf{R}}^1, \\ A\mathbf{e}^k = \sum_{l=1}^{k-1} B_l \mathbf{e}^l + \tau\widetilde{\mathbf{R}}^k, \quad k = 2, 3, \dots, N. \end{cases} \tag{19}$$

Take the inner product with $\mathbf{e}^1$ and $\mathbf{e}^k$ respectively, we have

$$\begin{cases} (A\mathbf{e}^1, \mathbf{e}^1) = \tau(\widetilde{\mathbf{R}}^1, \mathbf{e}^1), \\ (A\mathbf{e}^k, \mathbf{e}^k) = \sum_{l=1}^{k-1}(B_l\mathbf{e}^l, \mathbf{e}^k) + \tau(\widetilde{\mathbf{R}}^k, \mathbf{e}^k), \quad k = 2, 3, \dots, N. \end{cases} \tag{20}$$

For any symmetric matrix A, let $\{\lambda_i(A)\}_{i=1}^{M-1}$ denote the eigenvalues of the matrix A, $\lambda_{\min}(A)$ is the smallest eigenvalue and $\lambda_{\max}(A)$ is the largest eigenvalue among them, respectively. Note that $B_l(l = 0, \dots, k-1)$ are also symmetric positive definite because they are symmetric and diagonally dominated, with positive diagonal elements. For any symmetric matrix A, we can use the property of Rayleigh-Ritz ratio, i.e., for any vector $\mathbf{x} \in R^{M-1}, \mathbf{x} \neq \mathbf{0}$, we have (Theorem 4.2.2 in [24])

$$\lambda_{\min}(A) \le \frac{(A\mathbf{x},\mathbf{x})}{(\mathbf{x},\mathbf{x})} \le \lambda_{\max}(A).$$

Thus we obtain

$$\begin{aligned}
&(A\mathbf{e}^k,\mathbf{e}^k) \ge \lambda_{min}(A)(\mathbf{e}^k,\mathbf{e}^k) = \lambda_{min}(A)\|\mathbf{e}^k\|^2,\\
&|(B_l\mathbf{e}^l,\mathbf{e}^k)| \le (B_l\mathbf{e}^l,\mathbf{e}^l)^{1/2}(B_l\mathbf{e}^k,\mathbf{e}^k)^{1/2} \le \lambda_{max}(B_l)\|\mathbf{e}^l\|\|\mathbf{e}^k\|,\\
&|(\widetilde{\mathbf{R}}^k,\mathbf{e}^k)| \le \|\widetilde{\mathbf{R}}^k\|\|\mathbf{e}^k\|
\end{aligned}$$

where $\|\cdot\|$ stands for the Euclidean norm (or l_2 norm) on R^{M-1}, i.e., $\|\mathbf{v}\| = \left(\sum_{j=1}^{M-1} v_j^2\right)^{1/2}$. We arrive at

$$\begin{cases}
\|\mathbf{e}^1\| \le \frac{1}{\lambda_{min}(A)}\tau\|\widetilde{\mathbf{R}}^1\|,\\
\|\mathbf{e}^k\| \le \frac{1}{\lambda_{min}(A)}\left\{\sum_{l=1}^{k-1}\lambda_{max}(B_l)\|\mathbf{e}^l\| + \tau\|\widetilde{\mathbf{R}}^k\|\right\}, \quad k = 2,3,\ldots,N.
\end{cases} \tag{21}$$

We need to know the eigenvalues and eigenvectors for the matrices $A, \widetilde{B}_0$ and $B_l(l = 0,\ldots,k-1)$ above, all the matrices are tridiagonal ones. In fact, if we define

$$T = Tr(a,b,c) = \begin{pmatrix} b & c & 0 & \cdots & 0\\ a & b & c & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & a & b & c\\ 0 & \cdots & 0 & a & b \end{pmatrix}_{(M-1)\times(M-1)},$$

then the eigenvalues and associated eigenvectors are given by Thomas [22]. That is, we have eigenvalues

$$\lambda_j(T) = b + 2c\sqrt{\frac{a}{c}}\cos\frac{j\pi}{M}$$

and corresponding eigenvectors

$$\alpha_j = \begin{pmatrix} a_1\\ \vdots\\ a_k\\ \vdots\\ a_{M-1}\end{pmatrix}, \quad a_k = 2\left(\sqrt{\frac{a}{c}}\right)^k \sin\frac{kj\pi}{M}, \quad k = 1,\ldots,M-1$$

for $j = 1,\ldots,M-1$. Therefore, we have, for $j = 1,\ldots,M-1$,

$$\begin{aligned}
&\lambda_j(A) = 10 + 24\mu + 2(1-12\mu)\cos\frac{j\pi}{M} = 8 + 4\cos^2\frac{j\pi}{2M} + 48\mu\sin^2\frac{j\pi}{2M},\\
&\lambda_j(B_l) = 12\mu\lambda_{k-l}\left(-2 + 2\cos\frac{j\pi}{M}\right) = -48\mu\lambda_{k-l}\sin^2\frac{j\pi}{2M}, \quad l = 0,1,\ldots,k-2,\\
&\lambda_j(\widetilde{B}_0) = \lambda_j(B_{k-1}) = 10 - 24\mu\lambda_1 + 2(1+12\mu\lambda_1)\cos\frac{j\pi}{M} = 8 + 4\cos^2\frac{j\pi}{2M} - 48\mu\lambda_1\sin^2\frac{j\pi}{2M}.
\end{aligned}$$

From (21), we obtain

$$\|\mathbf{e}^k\| \le -\sum_{l=1}^{k-2} 6\mu\lambda_{k-l}\|\mathbf{e}^l\| + \|\mathbf{e}^{k-1}\| + \frac{1}{8}\tau(\|\widetilde{\mathbf{R}}^k\| + \|\widetilde{\mathbf{R}}^1\|), \quad k = 1,2\ldots,N. \tag{22}$$

By Lemma 4, we get

$$\|\mathbf{e}^k\| \le C\tau\sum_{l=1}^{k}\|\widetilde{\mathbf{R}}^l\|\exp\left(-6\mu\sum_{l=1}^{k-1}\lambda_{k-l} + 1\right) \le C\tau\sum_{l=1}^{k}\|\widetilde{\mathbf{R}}^l\|\exp(1+6\mu) \le C\tau\sum_{l=1}^{k}\|\widetilde{\mathbf{R}}^l\|.$$

Consequently, note that for uniform meshes, we have $\|\widetilde{\mathbf{R}}^k\|_{l^2} = h^{1/2}\|\widetilde{\mathbf{R}}^k\| = \mathcal{O}((k^{\gamma-1}\tau^{\gamma-1}+1)(\tau+h^4))$ and for $k\tau = t \le T$,

$$\tau\sum_{l=1}^{k} l^{\gamma-1}\tau^{\gamma-1} = \sum_{l=1}^{k}(l\tau)^{\gamma-1}\tau \le \int_0^t s^{\gamma-1}ds = \gamma^{-1}t^{\gamma} \le \gamma^{-1}T^{\gamma},$$

hence we get

$$\|\mathbf{e}^k\|_{l^2} \le C\tau \sum_{l=1}^{k} \|\widetilde{\mathbf{R}}^l\|_{l^2} \le C\tau(\tau + h^4)\sum_{l=1}^{k}(l^{\gamma-1}\tau^{\gamma-1} + 1) \le C(\gamma^{-1}T^{\gamma} + T)(\tau + h^4) \le C(\tau + h^4). \quad (23)$$

This completes the proof. □

5. Numerical experiments

Example 1. We consider the Example 1 given in paper [7],

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u}{\partial x^2}\right] + e^x\left[(1+\gamma)t^{\gamma} - \frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)}t^{2\gamma}\right], \quad 0 < t \le 1, \quad 0 < x < 1, \\ u(0,t) = t^{1+\gamma}, \quad u(1,t) = et^{1+\gamma}, \quad 0 \le t \le 1, \\ u(x,0) = 0, \quad 0 \le x \le 1, \end{cases} \quad (24)$$

with the exact solution being $u(x,t) = e^x t^{1+\gamma}$.

Since the l^2 error of our difference scheme is of first order in time and fourth order in space, assume that the error of our scheme satisfies (where $C_1(u)$ and $C_2(u)$ depend on the theoretical solution u)

$$\|e^k\|_{l^2} = \left(\sum_{j=1}^{M-1} |e_j^k|^2 h\right)^{1/2} = C_1(u)\tau + C_2(u)h^4,$$

then if we decrease the mesh size of h to $h/2$ and τ to $\tau/16$, then we get a factor $1/16$ for the error $\|e^k\|_{l^2}$. Therefore, to test the order of convergence of the scheme (10), we first let $h = \tau = 1/4$, then we let $h = 1/8, \tau = 1/64$, and $h = 1/16$ and $\tau = 1/1024$ (corresponding to upper half of Tables 1 and 2). We also choose another pair of the mesh sizes, starting from $h = \tau = 1/8$ (corresponding to lower half of Tables 1 and 2).

The results for $\|e\|_{l^2}$ and $\|e\|_{l^\infty} = \max_{1\le j\le M-1}|U_j^N - u(x_j, 1)|$ for the final time $t = 1$ (we abbreviate the superscript N), together with the relative error in brackets, and the experimental convergence order are shown in Tables 1 and 2 for $\gamma = 0.25$ and $\gamma = 0.75$, respectively.

In these tables, the experimental order of convergence $r(\tau, h)$ is computed by the formula

$$r(\tau, h) = \log_2(\|e(16\tau, 2h)\|_*/\|e(\tau, h)\|_*),$$

and $\|e(\tau, h)\|_*$ means the error $\|e\|_*$ computed with mesh sizes τ and h, with $\|e\|_* = \|e\|_{l^\infty}, \|e\|_{l^2}$ or $\|e\|_{l^2}/\|u\|_{l^2}$. For example, the order of convergence are obtained by data from the first line divided by the second line, and the second divided by the third, respectively, i.e., we get the results for $r(\frac{1}{64}, \frac{1}{8})$ and $r(\frac{1}{1024}, \frac{1}{16})$, respectively.

These results are in accordance with the order of convergence of our implicit difference approximation scheme. The numerical solution U were plotted using $h = \frac{1}{16}, \tau = \frac{1}{1024}$, with $\gamma = 0.25$ and $\gamma = 0.75$, respectively, in Figs. 1 and 2.

Table 1
Error and experiment order of convergence with $\gamma = 0.25$.

	$\|e\|_{l^\infty}$	Order	$\|e\|_{l^2}$	Order	$\|e\|_{l_2}/\|u\|_{l^2}$	Order
$h = \tau = \frac{1}{4}$	0.0148	–	0.0109	–	0.0073	–
$h = \frac{1}{8}, \tau = \frac{1}{64}$	0.0010	3.8875	7.7033e−4	3.8227	4.6995e−4	3.9573
$h = \frac{1}{16}, \tau = \frac{1}{1024}$	4.3249e−5	4.5312	3.1655e−5	4.6050	1.8472e−5	4.6691
$h = \tau = \frac{1}{8}$	0.0086	–	0.0063	–	0.0039	–
$h = \frac{1}{16}, \tau = \frac{1}{128}$	4.8554e−4	4.1467	3.5525e−4	4.1484	2.0731e−4	4.2336
$h = \frac{1}{32}, \tau = \frac{1}{2048}$	1.8928e−5	4.6810	1.3856e−5	4.6803	7.9151e−6	4.7110

Table 2
Error and experiment order of convergence with $\gamma = 0.75$.

	$\|e\|_{l^\infty}$	Order	$\|e\|_{l^2}$	Order	$\|e\|_{l_2}/\|u\|_{l^2}$	Order
$h = \tau = \frac{1}{4}$	0.0067	–	0.0048	–	0.0032	–
$h = \frac{1}{8}, \tau = \frac{1}{64}$	0.0016	2.0661	0.0012	2.0000	7.2406e−4	2.1439
$h = \frac{1}{16}, \tau = \frac{1}{1024}$	1.1182e−4	3.8388	8.1401e−5	3.8818	4.7502e−5	3.9300
$h = \tau = \frac{1}{8}$	0.0075	–	0.0054	–	0.0033	–
$h = \frac{1}{16}, \tau = \frac{1}{128}$	8.5296e−4	3.1363	6.2087e−4	3.1206	3.6231e−4	3.1872
$h = \frac{1}{32}, \tau = \frac{1}{2048}$	5.6363e−5	3.9197	4.0898e−5	3.9242	2.3363e−5	3.9549

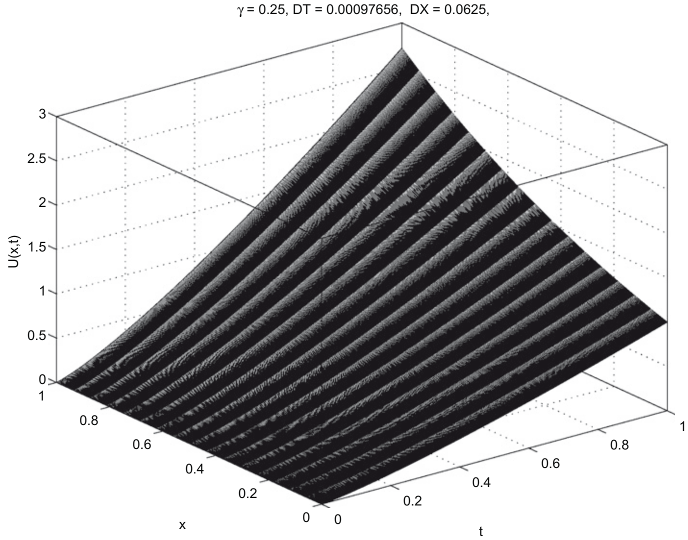

Fig. 1. Numerical solution $U(x_i, t^n), \gamma = 0.25$.

To increase the accuracy in time, we give the results with the Richardson extrapolation method given in [8] in Table 3, for the errors in l^2 and l^∞ norms at $t = 1$ (with the l^∞ norms in the brackets). For spatially fractional-order diffusion equations, temporally and spatially second-order accurate numerical estimates were obtained in Tadjeran and Meerschaert [25] by the classical Crank–Nicolson method combined with alternating direction implicit method and spatial extrapolation.

In Table 3, we can see that the Richardson extrapolation technique generally works well for increasing the order of local truncation error. We attribute the exceptions for $\gamma = 0.5$ and $\gamma = 0.6$ to that the errors are no longer decreased to halves so we tried another two pairs for $\tau = 1/2$ and $\tau = 1/4$, and the extrapolation is effective now.

Example 2. Another example is inspired by Example 1 given in paper [26],

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u}{\partial x^2}\right] + f(x,t), \quad 0 < t \le 1, \quad 0 < x < 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \le t \le 1, \\ u(x,0) = 0, \quad 0 \le x \le 1. \end{cases} \tag{25}$$

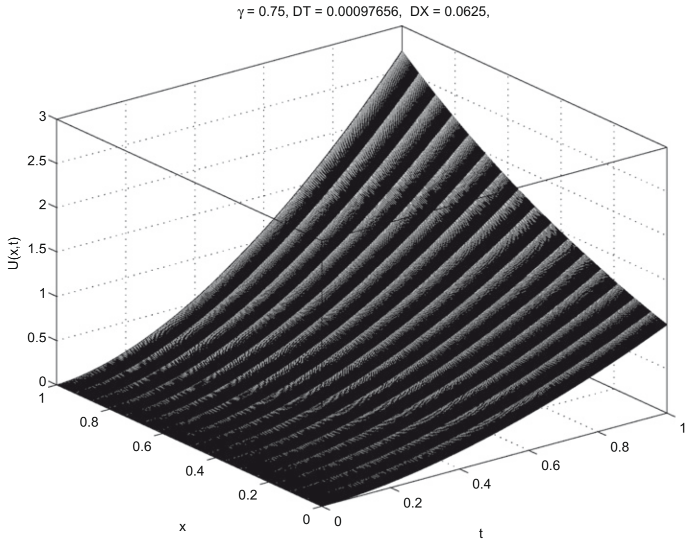

Fig. 2. Numerical solution $U(x_i, t^n), \gamma = 0.75$.

Table 3
Error in l^2 and l^∞ norms with $h = \frac{1}{64}$.

γ	$\tau = \frac{1}{8}$	$\tau = \frac{1}{16}$	After extrapolation
0.1	0.0040(0.0054)	0.0024(0.0033)	8.3460e−4(0.0011)
0.2	0.0059(0.0081)	0.0033(0.0045)	5.9432e−4(8.1364e−4)
0.3	0.0064(0.0087)	0.0031(0.0043)	7.4592e−5(1.0242e−4)
0.4	0.0055(0.0076)	0.0024(0.0032)	8.1669e−4(0.0011)
0.5	0.0036(0.0049)	0.0011(0.0015)	0.0014(0.0020)
0.6	7.1131e−4(9.5425e−4)	5.7619e−4(8.1066e−4)	0.0019(0.0026)
0.7	0.0032(0.0044)	0.0026(0.0036)	0.0020(0.0028)
0.8	0.0080(0.0110)	0.0050(0.0069)	0.0020(0.0027)
0.9	0.0138(0.0190)	0.0078(0.0107)	0.0018(0.0024)
γ	$\tau = \frac{1}{2}$	$\tau = \frac{1}{4}$	After extrapolation
0.5	0.0171(0.0234)	0.0096(0.0131)	0.0020(0.0027)
0.6	0.0139(0.0190)	0.0055(0.0075)	0.0029(0.0039)

Table 4
Error and the experimental order of convergence.

t	$h = \frac{1}{4},\ \tau = \frac{1}{16}$	$h = \frac{1}{8},\ \tau = \frac{1}{256}$	Order	$h = \frac{1}{16},\ \tau = \frac{1}{4096}$	Order
1/16	0.0016(0.5895)	4.5274e−4(0.1639)	1.8213	2.7829e−5(0.0101)	4.0240
1/8	0.0130(1.1787)	9.0399e−4(0.0818)	3.8461	5.5819e−5(0.0051)	4.0175
3/16	0.0218(0.8789)	0.0014(0.0548)	3.9608	8.4276e−5(0.0034)	4.0542
1/4	0.0302(0.6834)	0.0018(0.0413)	4.0685	1.1323e−4(0.0026)	3.9907
5/16	0.0385(0.5569)	0.0023(0.0333)	4.0652	1.4271e−4(0.0021)	4.0105
3/8	0.0467(0.4699)	0.0028(0.0280)	4.0599	1.7270e−4(0.0017)	4.0191
7/16	0.0551(0.4068)	0.0033(0.0242)	4.0615	2.0323e−4(0.0015)	4.0213
1/2	0.0635(0.3592)	0.0038(0.0213)	4.0627	2.3428e−4(0.0013)	4.0197
9/16	0.0721(0.3221)	0.0043(0.0191)	4.0676	2.6587e−4(0.0012)	4.0155
5/8	0.0807(0.2923)	0.0048(0.0174)	4.0715	2.9799e−4(0.0011)	4.0097
11/16	0.0895(0.2679)	0.0053(0.0159)	4.0778	3.3064e−4(9.8930e−4)	4.0027
3/4	0.0985(0.2476)	0.0059(0.0147)	4.0613	3.6384e−4(9.1474e−4)	4.0193
13/16	0.1076(0.2304)	0.0064(0.0137)	4.0715	3.9757e−4(8.5169e−4)	4.0088
7/8	0.1168(0.2157)	0.0069(0.0128)	4.0813	4.3184e−4(7.9766e−4)	3.9980
15/16	0.1262(0.2030)	0.0075(0.0121)	4.0727	4.6665e−4(7.5087e−4)	4.0065
1	0.1357(0.1919)	0.0081(0.0114)	4.0664	5.0200e−4(7.0993e−4)	4.0122

The exact solution of the problem (25) is $u(x,t) = t^2\sin(2\pi x)$, and $f(x,t) = \left(2t + \frac{8\pi^2 t^{1+\gamma}}{\Gamma(2+\gamma)}\right)\sin(2\pi x)$. In the numerical computations we let $\gamma = 0.5$.

As before, the l^2 error $\|e^k\|_{l^2}$ (with $t_k = k\tau$), together with the relative errors in brackets, and the experimental convergence order are shown in Table 4. We can see that the experimental convergence order is approximately 4, that is, the results nearly meet our anticipations.

Though the coefficient matrix of the unknowns is tridiagonal and the scheme can be easily solved by the Thomas algorithm, we must admit that since all time history must be in memory, the memory requirements are costly, and the computer memory will limit the step sizes. In fact, we have tried to use the "Short Memory" principle as proposed in [2], but the numerical experiments are not very satisfactory unless sufficiently many previous time steps have been included. The author thanks one referee for pointing out this comment.

6. Conclusion

In this paper we have investigated the stability and accuracy of a compact implicit difference scheme for solving the fractional diffusion equation. This compact implicit difference scheme has the advantage of high accuracy with the coefficient matrix still being a tridiagonal one, therefore, the linear system of equations are easy to solve. We have proved that the method is unconditionally stable for $0 < \gamma < 1$. We have also shown that the method has accuracy of four in the spatial grid size and one in the fractional time step. The conclusions are verified by some numerical experiments.

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